# Mortar method using bi-orthogonal nodal functions applied to A-φ formulation

M. Aubertin[1], T. Henneron[2], F. Piriou[2], J-C. Mipo[1]
[1]Valeo Systèmes Electriques 2 rue André Boulle, BP 150, 94017 Créteil France
[2]L2EP, Université Lille 1, Bât. P2, 59655 Villeneuve d'Ascq, France,
E-mail: thomas.henneron@univ-lille1.fr

**Abstract — For magnetodynamic problem decomposed into subdomains, the Mortar method can be associated with potential formulations to connect nonconforming meshes. In this paper, we propose to use the bi-orthogonal nodal shape functions with the Mortar method in the case of A-φ formulation. An academic example will be studied to shown the accuracy of the proposed model.**

## I. Introduction

To model electromagnetic devices, the finite element method is used. In order to minimize the mesh, numerical methods based on the domain decomposition can be used. The studied system is divided into several sub-domains, according to their dimensions and their sizes. Each sub-domain is discretized independently. The Mortar method can be used to connect the solution of different non-conforming meshes. In the case of magnetodynamic problem, the electric **A**-φ formulation can be used. From the Mortar method, a relation associated with each potential must be defined. These relations are introduced in order to verify the continuity of the fields. The drawback of this approach is the inversion of submatrixes associated with the connection of the solution in non-conforming meshes. In order to avoid this constraint, the bi-orthogonal shape functions can be used [1, 2].

In this communication, we propose to investigate the use of the bi-orthogonal shape functions with the Mortar method and the **A**-φ formulation. The numerical model is presented and an academic example is analyzed.

## II. Numerical Model

### A. *A- φ formulation*

Let us consider a domain D of boundary Γ (Fig. 1). D is divided into two subdomains $D_1$ and $D_2$ of boundary $\Gamma_1$ and $\Gamma_2$ respectively ($D=D_1 \cup D_2$). Both subdomains are separated by a boundary denoted $\Gamma_R$. In D, a conducting part $D_C$ of boundary $\Gamma_C$ belonging to $D_1$ and $D_2$ through by $\Gamma_R$ is considered.

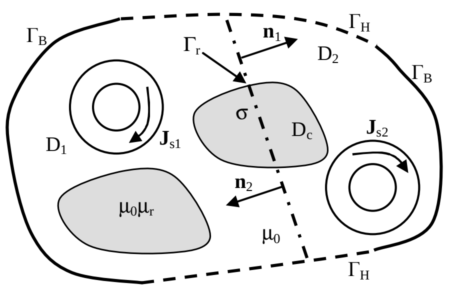


Fig. 1: Studied domain

Using the formulation in term of vector potential **A** and scalar potential φ, the weak formulation to solve in each subdomain k can be written such that:

$$\begin{aligned}&(\mu^{-1}\mathbf{curlA}_k, \mathbf{curlA}'_k)_{D_k} + (\sigma\partial_t \mathbf{A}_k, \mathbf{A}'_k)_{D_{ck}} + (\sigma \mathbf{grad}\varphi_k, \mathbf{A}'_k)_{D_{ck}} \\ &+ \langle \mu^{-1}(\mathbf{curlA}_k + \mathbf{N}\Phi)\wedge \mathbf{n}_1, \mathbf{A}'_k\rangle_{\Gamma_k} \\ &= (\mathbf{J}_{sk}, \mathbf{A}'_k)_{D_k} - (\mu^{-1}\mathbf{N}\Phi, \mathbf{curlA}'_k)_{D_k} - (\sigma\partial_t \mathbf{K}\Phi, \mathbf{A}'_k)_{D_{ck}}\end{aligned} \quad (1)$$

$$\begin{aligned}&(\sigma\partial_t\mathbf{A}_k, \mathbf{grad}\varphi'_k)_{D_{ck}} + (\sigma\mathbf{grad}\varphi_k, \mathbf{grad}\varphi'_k)_{D_{ck}} \\ &- \langle\sigma(\partial_t\mathbf{A}_k + \partial_t\mathbf{K}\Phi + \mathbf{grad}\varphi_k).\mathbf{n}_k, \varphi'_k\rangle_{\Gamma_{ck}} = -(\sigma\partial_t\mathbf{N}\Phi, \mathbf{grad}\varphi'_k)_{D_{ck}}\end{aligned} \quad (2)$$

with Φ a magnetic flux, **N** and **K** are the source fields (**curl K**= **N**) associated with Φ [3], $\mathbf{J}_{s1}$ and $\mathbf{J}_{s2}$ the current density in two inductors. In your development, the components of **N** and **K** are equal to zero on $\Gamma_r$. $(.,.)_{Dk}$ indicates the scalar product on the domain $D_k$, $<.,.>_{\Gamma k}$ the scalar product on $\Gamma_k$ and $\partial_t$ the time derivative. In these expressions, $\mathbf{A}'_k$ and $\varphi'_k$ represent the test functions which are chosen in the same discrete space that the shape functions of **A** and φ respectively. Taking into account the boundary conditions and the properties of $\mathbf{A}'_k$ in (1), the surface integral term on $\Gamma_k$ which corresponds to the tangential component of the magnetic field can be reduce to $\Gamma_r$. In the same way, in (2), the surface integral term which corresponds to the normal component of the current density can be reduced on $\Gamma_{rc}$.

### B. *Continuity at the interface $\Gamma_R$*

To ensure the continuity of the fields at the interface $\Gamma_R$, the classical next conditions must be verified:

$$\mathbf{H}_1 \wedge \mathbf{n}_1\big|_{\Gamma_r} = -\mathbf{H}_2 \wedge \mathbf{n}_2\big|_{\Gamma_r} = \mathbf{H}_t \quad (3)$$

$$\mathbf{J}_1.\mathbf{n}_1\big|_{\Gamma_{rc}} = -\mathbf{J}_2.\mathbf{n}_2\big|_{\Gamma_{rc}} = \mathbf{J}_n \quad (4)$$

where $\mathbf{J}_1$ and $\mathbf{J}_2$ the eddy current in $D_C$. Moreover, we also must verify the continuity of both potentials on $\Gamma_r$ such that:

$$\langle \mathbf{A}_1 - \mathbf{A}_2, \mathbf{A}'\rangle_{\Gamma_r} = 0 \quad \text{and} \quad \langle \varphi_1 - \varphi_2, \varphi'\rangle_{\Gamma_{rc}} = 0 . \quad (5)$$

with **A'** and φ' test functions so that the choice is introduced in the next section.

### C. *Discrete form*

At this step of the analysis, we must discretize the potentials $\mathbf{A}_k$ and $\varphi_k$, the fields $\mathbf{H}_t$ and $\mathbf{J}_n$ and define the test functions $\mathbf{A}'_k$, $\varphi'_k$, **A'** and φ'. Using Witney's element the vector potential $\mathbf{A}_k$ is naturally discretized in the edge element space and the scalar potential $\varphi_k$ in the nodal element space

[4]. Consequently, using the Galerkin method, $\mathbf{A}'_k$ and $\varphi'_k$ take the form of edge and nodal elements respectively. On $\Gamma_r$, the tangential component of magnetic field $\mathbf{H}_t$ and the normal component of the induced current $\mathbf{J}_n$ belonging to $\Gamma_{rc}$ are respectively discretized in the edge and nodal element space. At last, the test functions $\mathbf{A}'$ and $\varphi'$ on $\Gamma_r$ are taken in the same space than $\mathbf{H}_t$ and $\mathbf{J}_n$ respectively. In these conditions, the discrete form of (5-a) and (5-b) are written under the matrix form:

$$C_e A_{1\Gamma r} = D_e A_{2\Gamma r} \quad \text{and} \quad C_n \varphi_{1\Gamma r} = D_n \varphi_{2\Gamma r} \qquad (6)$$

where $A_{1\Gamma r}$ and $A_{2\Gamma r}$ represent the vector of the circulations of the vector potential on $\Gamma_r$ and $\varphi_{1\Gamma r}$ and $\varphi_{2\Gamma r}$ the vector of nodal values of the scalar potential on $\Gamma_{rc}$. The elementary terms of the matrix $C_e$ and $D_e$ take the form:

$$c_{ei,j} = \left\langle w'_{ei}, w_{ej} \right\rangle_{\Gamma_{rc}} \quad \text{and} \quad d_{ei,k} = \left\langle w'_{ei}, w_{ek} \right\rangle_{\Gamma_{rc}} \qquad (7)$$

with $\mathbf{w}_{ej}$ and $\mathbf{w}_{ek}$ the edge shape functions associated with $A_1$ and $A_2$ respectively and $\mathbf{w}'_{ei}$ the test function. The matrix $C_n$ and $D_n$ are defined similarly with nodal shape functions. In (5-a) and (5-b), the integral surface term on $\Gamma_r$, written in discrete form, depend on the same shape functions than the discrete continuity relations on (6). Consequently, they can be written:

$$F_{H1\Gamma r} = C^t_e H_{\Gamma r} \quad \text{and} \quad F_{H2\Gamma r} = D^t_e H_{\Gamma r} \qquad (8)$$

$$F_{J1\Gamma r} = C^t_n J_{\Gamma r} \quad \text{and} \quad F_{J2\Gamma r} = D^t_n J_{\Gamma r} \qquad (9)$$

At this step of the analysis, using the Mortar method, it is possible to substitute the unknowns $A_{1\Gamma R}$ and $\varphi_{1\Gamma R}$ and combining FE equations system such that it is not necessary to calculate $H_{\Gamma r}$ and $J_{\Gamma r}$. Nevertheless, we must compute the inverse of the matrixes $C_e$ and $C_n$. To simplify theirs computations, it is possible to define bi-orthogonal shape functions for $\mathbf{w}'_e$ and $w'_n$ [1, 2]. In these conditions, the matrix $C_e$ and $C_n$ begin diagonal. It is possible to find bi-orthogonal shape functions for nodal element and classical edge element. Unfortunately, it is known that the classical edge element used in Mortar method induced numerical error on the boundary of non-conforming mesh. To avoid this difficulty, a second family of edge element is classically used [5]. Consequently, for edge elements, we propose to build the test functions $\mathbf{w}'_e$ from bi-orthogonal nodal function $w'_n$. In this case, we have:

$$\mathbf{w}'_{ei,j} = w'_{ni} \, \mathbf{grad} w'_{nj} \qquad . \qquad (10)$$

The matrix $C_e$ is not diagonal but the number of terms is notably reduced.

## III. Application

The proposed approach has been used to model a conducting hollow sphere crossed by a sinusoidal flux density. The conductivity is equal to $10^7 (\Omega m)^{-1}$ and the maximal value of magnetic flux density to 1T. To study the system, two meshes have been considered. The first one M1 is fine and the second M2 is composed with a part of M1 and a coarse complementary mesh (Fig. 2).

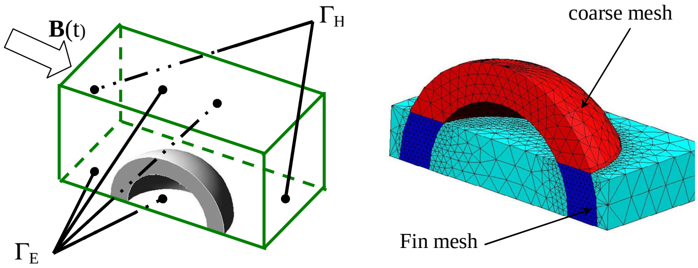


Fig. 2: Studied example (2.a) and part of the mesh M2 (2.b)

In Fig. 2, we can see that the boundary $\Gamma_R$ crosses horizontally the sphere. With the three meshes, the modeling has been done for a frequency of 100Hz. In Fig. 3, the losses powers in the sphere in function of time are presented. We can observe that the results obtained from M1 and M2 meshes give the same results.

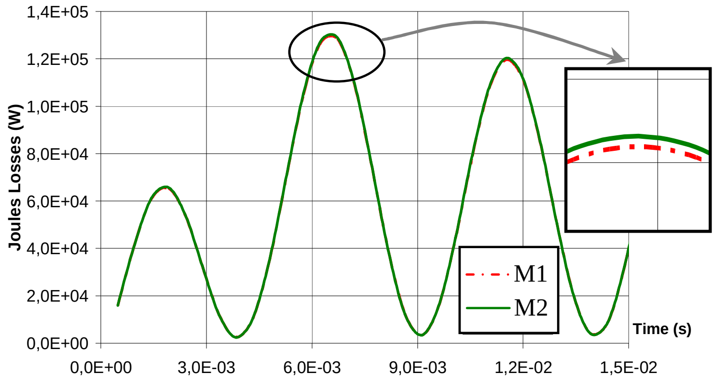


Fig. 3: Losses powers in the sphere with fine and non-conforming meshes

## IV. Conclusion

In this paper, the Mortar method has been used to connect non-conforming meshes in magnetodynamic problem. To reduce the memory space, the bi-orthogonal nodal functions have been used. For the edge functions, we propose to define the second family built from bi-orthogonal nodal functions. As example of application, a hollow sphere is studied and the obtained results with a non-conforming mesh are in good agreement compared with a fine mesh. The method can be applied to the magnetic **T-Ω** formulation.

## V. References

[1] B. Wohlmuth, "Mortar finite element method using dual spaces for the Lagrange multiplier", SIAM, Vol. 38, No 3, pp. 989-1012, 2000.

[2] E. Lange, F. Henrotte and K. Hameyer, "A variationnal Formulation for Nonconforming Sliding Interfaces in Finite Element Analysis of Electric Machines", *IEEE Trans. on Magn.*, vol.46, no.8, pp. 2755-2758, 2010.

[3] P.Dular, J. Gyselinck, T. Henneron, F. Piriou, "Dual Finite Element Formulations for Lumped Reluctances Coupling", *IEEE Trans. on Magnetics*, 41(5): 1396– 1399, 2005.

[4] A. Bossavit, "A rationale for edge-elements in 3-D fields computations", *IEEE Trans. Mag.* vol. 24, pp. 74–79, Jan. 1988.

[5] F. Rappetti Y. Maday, F. Bouillault and A. Razek, "Eddy-current calculations in three-dimensionnal moving structures", *IEEE Trans. on Magn.*, vol.38, n°2, pp. 613-616, 2002.